

New results for the quantum supersymmetric kink

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Abstract

We review our work on computations of the quantum corrections to the mass and the central charge of the susy kink. For the mass corrections, we find that the widely used momentum cut-off scheme gives an incorrect result, but we deduce through smoothing of the cut-off an extra term in the mass formula, which produces the correct result. We discover the importance of boundary effects for the mode number cut-off regularization scheme. We introduce the notion of delocalized boundary energy. We discuss two discrete Z_2 symmetries and their importance to the mode number approach. For the central charge corrections, we use momentum cut-off regularization with two cut-offs, one for propagators and another for Dirac delta functions. We then compute the quantum anomaly in the central charge, and find that it restores the BPS bound at the one-loop level if the two cut-offs are equal.

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The problem of quantum corrections to the mass of the bosonic and susy kink was intensively studied in many papers in the 1970's and 1980's (see [1] and references therein). In 1997, with the advent of extended particles in string theory, the issue of quantum corrections to extended objects, and the issue of the BPS bound at the quantum level, was brought back into focus in ref. [2]. This paper emphasized the contradictory results coming from the two major regularization approaches, namely the momentum and mode number cut-off schemes. Another issue emphasized in [2] was that the BPS bound did not seem saturated at the quantum level, though clearly it holds classically. Given the discrepancies between the results of these two schemes, in [3] another regularization scheme for the evaluation of the quantum mass of a soliton was developed, which we have called derivative regularization. According to this method one first evaluates the sums $\frac{\partial}{\partial m} \sum \omega_n$ which are better convergent than $\sum \omega_n$. In [3] it was also proposed that there is an anomaly which is responsible for the BPS bound saturation on the quantum level. The results of [3] on the quantum mass of a (susy) kink were confirmed by the phase shift approach of [4] who also presented a calculation of the central charge. The anomaly was discovered later by [5] and with this anomaly the saturation of the BPS bound was explained. These new results did not answer the question “what was wrong with the old schemes ?” Recently we reanalyzed in [6, 7, 8] the old schemes. In this contribution we present the results of this analysis and show that these schemes require some modifications. These modifications have a physical motivation behind them and lead to correct results.

We study the (susy) kink with the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}U^2(\phi) - \frac{1}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{1}{2}c\frac{dU}{d\phi}\bar{\psi}\psi \quad (1)$$

where $U[\phi(x)] = \sqrt{\frac{\lambda}{2}}\left(\phi^2 - \frac{\mu_0^2}{\lambda}\right)$ and $\bar{\psi} = \psi^\dagger i\gamma^0$. The field equations allow the static solution $\phi(x) = \frac{\mu}{\sqrt{\lambda}} \tanh \frac{\mu x}{\sqrt{2}}$, $\psi = 0$, the kink, see fig.1. To discretize the spectrum of fluctuations we introduce boundaries. As we will describe later, there is usually extra energy created by the boundaries. One needs to subtract this energy in order to obtain the correct mass of the (susy) kink [9, 5].

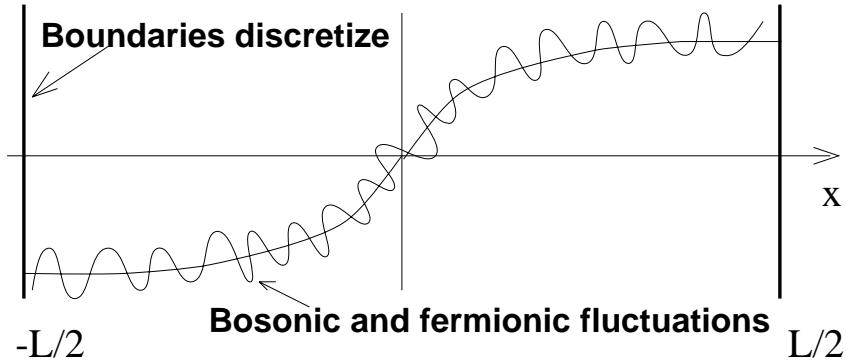


Fig1. The kink solution and quantum fluctuations around it.

The kink solution satisfies the BPS equation $\partial_x \phi + U = 0$. One may define the central charge from the algebra of the susy charges, and it is well known that the BPS bound is saturated at the classical level. To compute the quantum one-loop corrections we first renormalize the theory by replacing μ_0^2 by $\mu^2 + \delta\mu^2$, where the mass counter term $\delta\mu^2$ in the susy case is fixed by requiring that it cancels the tadpole graphs

$$(\delta\mu^2)_{susy} = \frac{\lambda\hbar}{4\pi} \int_{-\Lambda}^{\Lambda} \frac{dk}{\sqrt{k^2 + m^2}}; \quad \text{tadpole} + \text{tadpole} + \text{tadpole} = 0 \quad (2)$$

If one plugs the expression for the kink background into the renormalized Lagrangian, one obtains a counterterm of order \hbar , namely $\Delta M = \frac{m}{\lambda} (\delta\mu^2)_{susy}$. Thus if we define the mass of the kink as

$$M = \langle \text{kink} | \hat{H} | \text{kink} \rangle - \langle \text{trivial} | \hat{H} | \text{trivial} \rangle \quad (3)$$

then the expression for the one-loop correction to the mass of the susy kink is

$$M^{(1)} = \frac{\hbar}{2} \left(\sum [\omega_n^b - \omega_n^{b,(0)}] \right) - \frac{\hbar}{2} \left(\sum [\omega_n^f - \omega_n^{f,(0)}] \right) + \Delta M \quad (4)$$

The computation of this expression is rather tricky. The first correct method to obtain the one loop mass correction is to begin with a kink-antikink background, and to compute the spectrum of fluctuations around this background. By iterating the Dirac equation for $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, all field equations become Schorödinger equations with the potentials shown in figure 2.

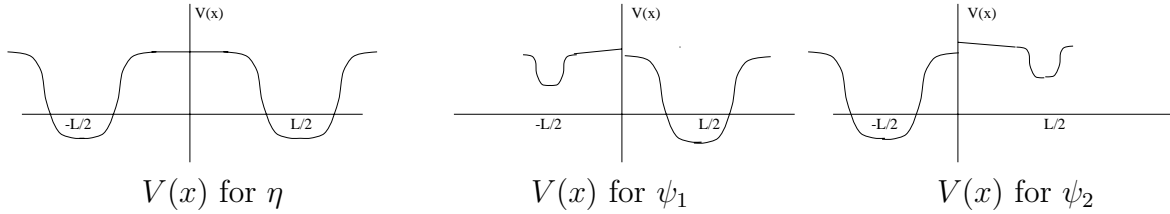


Figure 2. *Potentials for the bosonic and fermionic fluctuations in the kink-antikink system.*

Suppose one starts with the constant solution $\phi_{cl} = \mu/\sqrt{\lambda}$, and starts pulling the field ϕ near zero down to $\phi_{cl} = -\mu/\sqrt{\lambda}$. One creates in this way a kink-antikink system with the same boundary conditions in the kink-antikink sector as in the trivial sector, and the mass of the kink is then one-half the mass of the kink-antikink system [9]. The results are independent of the choice of boundary conditions. A correct method for a single kink is the derivative regularization of [3]. The idea is to take $\frac{\partial}{\partial m} M^{(1)}$ and then, after computing the sums, to fix all ambiguities by imposing the renormalization condition $M^{(1)}(m=0) = 0$ and to integrate to get $M^{(1)} = m \frac{\partial}{\partial m} M^{(1)}$. Again we must use the same boundary conditions in both sectors. The answers obtained by this method are

$$M^{(1)}(bos) = -m\hbar \left(\frac{3}{2\pi} - \frac{\sqrt{3}}{12} \right) \quad M^{(1)}(susy) = -\frac{m\hbar}{2\pi} \quad (5)$$

This approach gives the correct result whenever the boundary conditions lead to a finite result for (4), but it does not explain what was wrong with the other approaches.

The problems of the momentum cut-off were solved recently in [6]. We introduced a smooth function $f(k, \Lambda) \sim \Theta(\Lambda - k)$ into the quantization conditions:

$$kL + f(k, \Lambda)\delta(k) = 2\pi n \quad (6)$$

where the phase shifts $\delta(k)$ are due to the reflectionless kink potential⁴ for the fluctuations. The purpose of $f(k, \Lambda)$ is to make a smooth transition between the nontrivial quantization conditions at small k to the trivial conditions for $k > \Lambda$. This modification leads to new terms in the expression for the density

$$\frac{dn}{dk} = \frac{L}{2\pi} + f(k, \Lambda)\delta'(k) + f'(k, \Lambda)\delta(k) \quad (7)$$

The term $f\delta'(k)$ truncates the infinite sums in a natural way, and this kind of truncation is widely used in the Casimir effect. The last term is absent in the naive approach and it is this term which is responsible for obtaining the correct answer (which follows from using the spectral density (7) to transform the sums of (4) into integrals).

Once the traditional momentum cut-off for the mass corrections was rehabilitated, we turned [6, 7] to the direct computation of the central charge anomaly with momentum cut-off regularization. The classical Noether current for supersymmetry is

$$j^\mu = -(\partial_\nu \phi)\gamma^\nu \gamma^\mu \psi - U\gamma^\mu \psi \quad (8)$$

By using equal-time canonical (anti) commutation relations we can compute Z from

$$\{Q^\alpha, \bar{Q}_\beta\} = 2i(\gamma^\nu)^\alpha{}_\beta P_\nu - 2i(\gamma^5)^\alpha{}_\beta Z \quad (9)$$

The expression for the central charge is

$$Z_{bos} = \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dx' \left\{ U[\phi(x)]\partial_{x'}\phi(x') + U'[\phi(x)]\bar{\psi}(x')\gamma^1\psi(x) \right\} \hat{\delta}(x - x') \quad (10)$$

Formally the first term is a total derivative and the second term vanishes because ψ is a Majorana spinor, but one should use regularization. To regulate this expression we impose a momentum cut-off Λ on the propagators, and a cut-off K on the Dirac delta functions which follow from the equal-time canonical commutation relations, where

$$\langle \eta(x)\eta(x') \rangle = \int_{-\Lambda}^{\Lambda} \frac{dq}{2\pi} \frac{e^{iq(x-x')}}{\sqrt{q^2 + m^2}}; \quad \hat{\delta}(x) = \int_{-K}^K \frac{dq}{2\pi} \exp(iqx) \quad (11)$$

Expanding ϕ into $\phi_K + \eta$ where η is the quantum field, the terms quadratic in η are of the form $\frac{1}{2}U''\langle \eta^2(x) \rangle \partial_{x'}\phi_K(x') + U'\langle \eta(x)\partial_{x'}\eta(x') \rangle$. The former is canceled by the mass renormalization whereas the latter is a typical anomaly effect: it is formally zero by symmetric integration if

⁴The kink model is member of a whole class of reflectionless potentials which share many special properties. For a nice discussion see [4].

one uses the unregulated delta function, but with the truncated delta function one produces a kind of point splitting and the contribution is nonzero and finite. Shifman, Vainshtein and Voloshin [5] have shown that there is indeed an anomaly in the current multiplet of which Z is part.

Taking the vacuum expectation value of the regulated central charge we can easily obtain the quantum correction to Z . We found that it only depends whether $\Lambda > K$, $\Lambda = K$ or $\Lambda < K$. Physically one needs $\Lambda = K$ because $(\square + m^2)\hat{G}(x, x') = \hat{\delta}(x - x')$ where $\hat{G}(x, x')$ is the regulated propagator. The propagator in the kink background can be written as a sum of the flat-space propagator, the propagator with one insertion of the background, two insertions etc., see figure 3.

$$\begin{aligned}
 & \left(\text{Loop with } x \text{ and arrow} + \text{Loop with 1 wavy line} + \text{Loop with 2 wavy lines} + \dots \right) - \left(\text{Loop} \right) \\
 & = \text{Loop with 1 wavy line} + \text{Loop with 2 wavy lines} + \text{Loop with 3 wavy lines}
 \end{aligned}$$

Figure 3. *The insertions in the loops are due to the kink background.*

We found that only the flat space propagator gives a nonzero result, confirming that one is dealing with an anomaly which is a local effect independent of the background. For $\Lambda = K$ we found that the BPS bound is saturated at one loop [6]. [In [5] the local Z density and energy density around the kink were studied.]

The BPS bound should be saturated at the quantum level if multiplet shortening occurs. Initially it was believed that irreducible non-BPS multiplets for $N = (1, 1)$ susy in $1 + 1$ dimensions (with $H \neq Z$) contained 4 states and BPS multiplets (with $H = Z$) only 2 states. Then it was noted in [6] that non-BPS multiplets contain only 2 states. Finally, [5] and later [8] showed that BPS multiplets contain only one state. This state is of the form $(1 + c_0)|\Omega\rangle$ where c_0 is the zero mode of the 2-component Majorana fermion in a kink background. It may seem strange to have a state which is half bosonic and half fermionic, but note that in $1 + 1$ dimensions the distinction between bosons and fermions is less clear. Since there is multiplet shortening, after all, one should find that $\langle H \rangle = M$ equals $\langle Z \rangle$ and that is now indeed the case. Though the BPS multiplet is really one dimensional, there exist two gauge copies of it, connected by the discrete Z_2 gauge symmetry [8]. Our results are in accord with the results of [5, 10].

The two discrete Z_2 symmetries $\psi \rightarrow -\psi$, and $\phi \rightarrow -\phi$ with $\psi_1 \leftrightarrow \pm\psi_2$ play an important role in the computation of the quantum corrections by the mode number regularization scheme. We started with a kink-antikink system (see figure 4), following [9].

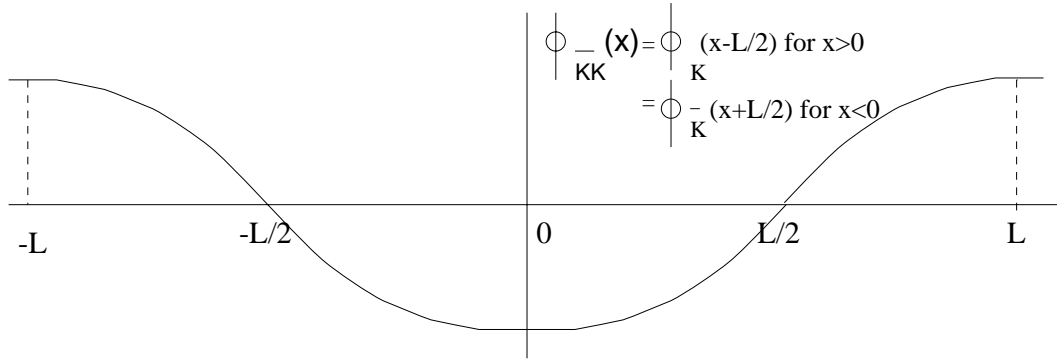


Figure 4. *The kink-antikink configuration; there is a cusp at $x = 0$.*

This system is topologically in the trivial sector (we have also developed in [8] a susy version of Manton and Samols' sphalerons-on-a-circle model, see [11], which lead to the same conclusions). One may use any (for example periodic) boundary conditions for the kink-antikink system and then deduce the set of boundary conditions needed for a single kink by looking at what happens in the middle.

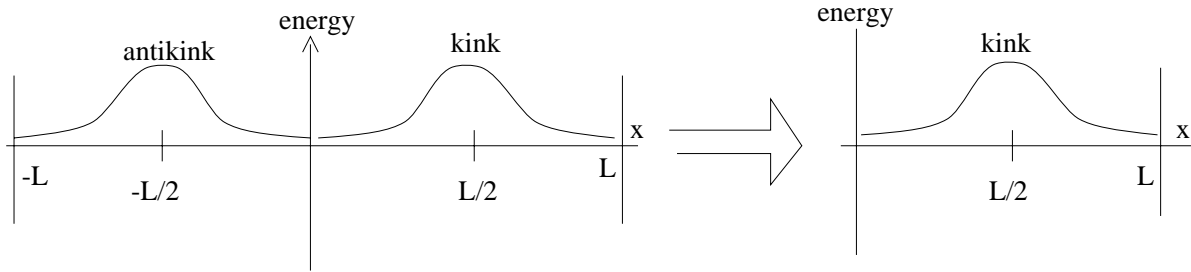


Figure 5. *The boundary conditions for a single kink can be deduced from the boundary conditions of the kink-antikink system.*

Then, if one averages over this set, one automatically obtains the correct answer. But why would it be necessary to average? It turns out [8] that for a fixed set of boundary conditions for a single kink (namely, the same boundary conditions in the kink sector as in the trivial sector) there is always a boundary energy, see figure 6.

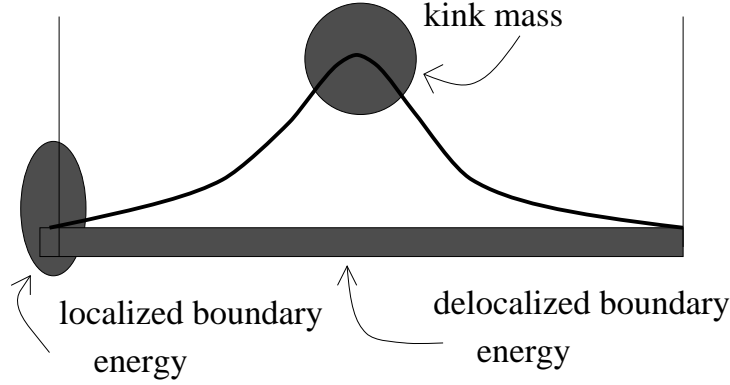


Figure 6. *The total energy density in the kink background.*

For certain boundary conditions, one obtains standing waves and then the boundary conditions are “visible” for the kink; then there is localized boundary energy in the kink sector. For other boundary conditions, one finds plane waves as solutions, and then the boundary conditions are “invisible” for the kink but there can be delocalized boundary energy. Boundary conditions which are invisible in the kink sector are visible in the trivial sector and vice-versa. For a fixed set of boundary conditions, making a twist (i.e. exchanging ψ_1 and ψ_2 and replacing $\phi \rightarrow -\phi$) at one side of the kink produces a current which drives half of a fermionic degree of freedom away from the kink to the boundary, see figure 7. This loss of half a degree of freedom was found to occur for Dirac fermions in [12] and explained the discovery of [13] that a kink carries one half unit of fermion charge.

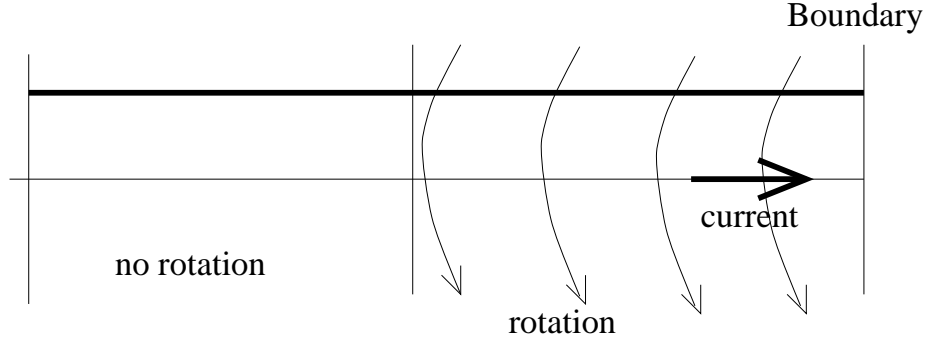


Figure 7. *The chiral rotation causes a current which leads to the accumulation of half the degree of freedom at the boundary.*

On the other hand, the Z_2 gauge symmetry guarantees that for a gauge invariant set of boundary conditions, such as, for example, the set “periodic+anti-periodic+twisted periodic+twisted anti-periodic,” all boundary energies cancel. The chiral rotation of figure 7 produces from, say, the periodic conditions in one sector, both the twisted periodic and twisted anti-periodic conditions in the other sector, which is due to the fact that one must complement the chiral rotation by a gauge transformation to keep our fermions real.

Thus we have not only rescued the oldest and most popular methods for computation

of quantum corrections, mode number and momentum regularization, but we have also discovered the importance of boundary effects for these computations. We have realized that there are two discrete Z_2 symmetries and that the failure to take a gauge invariant set of boundary conditions for the computation may even result in the appearance of delocalized boundary energy, which would mean the failure of the cluster decomposition principle.

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